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On the solutions of some linear operator non-polynomial differential equations

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Abstract. A method is given for solving a class of linear operator non-polynomial differential equations, i.e. equations with linear differential operators representing non-polynomial functions of the operator of differentiation with respect to the argument of the unknown function. Exact solutions of some particular differential equations of such type are obtained. As in a previous paper of ours it is also obtained that not all linear operator non-polynomial differential equations are of infinite order, i.e. some of them have only a finite number of linearly independent solutions, and this concerns some equations with a great importance in theoretical and mathematical physics.

1. Introduction

As is known, the theory of ordinary and partial differential equations, as well as theoretical and mathematical physics as a rule deal with and solve equations, each of which can be written in the form of an equality to zero of a certain function of the arguments of the unknown function, of that function itself and of its derivatives of all orders, lower than or equal to a definite finite order, the latter being the order of the differential equation (see, for instance, Goursat 1911, Ince 1926, Kamke 1959, Stepanov 1966, Petrovskii 1970, Courant 1962, Morse and Feshbach 1953, Murphy 1960, Babich *et al* 1964, Vladimirov 1976). Comparatively few papers until now have been devoted to infinite systems of differential equations (see, for example, Valeev and Zhautikov 1974), and with regard to the theory of linear difference equations Gel'fond (1951, 1967) considered a definite type of linear differential equations of infinite order with constant coefficients. However, in a number of fields of theoretical physics, such as the theory of solids, the relativistic quantum theory and others, one must often consider and solve linear equations with non-polynomial differential operators, i.e. operators representing given non-polynomial functions of the operators of differentiation with respect to the arguments of the unknown function. Besides, it is considered that the operators of such type have a non-local character, hence the solving of the equations is accompanied by serious mathematical difficulties.

In the present paper a method for solving linear operator non-polynomial differential equations is proposed. Some particular cases of such operator non-polynomial differential equations are of great interest in theoretical physics since they serve to describe important physical phenomena. It is also obtained that not all non-polynomial differential operators are non-local ones (Dimitrov 1981) and this concerns some operators with a great importance in theoretical and mathematical physics (see, for instance, Bjorken and Drell 1964, Ziman 1972).

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2. General consideration of a linear ordinary differential equation

Let $F(z)$ be an arbitrary given (in general multivalued) holomorphic function in the whole complex z plane, possibly, with the exception of a set of isolated singular points $z_j^{(0)}$, $j=1,2,3,\ldots$, which can be also branch ones. Then, consider an ordinary homogeneous differential equation of the type

$$L_x(y) \equiv [F(q_l(x)) + f(x) - a]y = 0 \tag{1}$$

where y is the unknown function of the real variable x, a is a constant, $F(q_l(x))$ is a differential operator which is obtained from $F(z)$ by the substitution of z with the linear (polynomial) differential operator of lth order $q_l(x)$†,

$$q_l(x) = D^l + g(x) \qquad D \equiv \mathrm{d}/\mathrm{d}x, \tag{2}$$

while $f(x)$ and $g(x)$ are given functions of x. We shall consider separately only the cases: (i) $g(x)=0$, $l=1$, and $f(x)$ is a rational function; (ii) $f(x)=0$, and $g(x)$ is an infinitely differentiable function.

The solutions of equation (1) in the case (i), which are defined in some interval I of the variable x, will be sought with the help of the Laplace transformation, i.e. in the form of the following integral representation (Goursat 1911, ch 20, Kamke 1959, part 1, ch 5, Morse and Feshbach 1953, ch 5)

$$y = \int_C \mathrm{d}\tau\, \chi(\tau)\, \mathrm{e}^{x\tau} \tag{3}$$

where C is a path of integration (independent of x) in the complex τ plane assuring the existence of the integral, while $\chi(\tau)$ is a non-zero in some neighbourhood G of the path C holomorphic function satisfying an equation representing the Laplace transformation of the equation (1). Let $f(x)$ have the form

$$f(x) = P_m(x)/Q_n(x) \tag{4}$$

where $P_m(x)$ and $Q_n(x)$ are polynomials of degrees m and n, respectively,

$$P_m(x) = \sum_{\mu=0}^{m} a_\mu x^\mu \qquad Q_n(x) = \sum_{\nu=0}^{n} b_\nu x^\nu. \tag{5}$$

Now, in order to find the equation for the function $\chi(\tau)$, the operator function $F(D)$ will be expressed as a series in terms of powers of the operator $D - z_0$ for a value z_0 ($z_0 \neq z_j^{(0)}$, $j=1,2,3,\ldots$) of the complex variable z, so that we shall write

$$F(D) = \sum_{s=0}^{\infty} A_s (D - z_0)^s \qquad A_s = \frac{1}{s!}\left[\frac{\mathrm{d}^s F(z)}{\mathrm{d}z^s}\right]_{z=z_0}. \tag{6}$$

Thus, with the help of the expressions (4) and (5) and the representation (6) in the considered case the equation (1) is written in the form

$$L_x(y) \equiv \sum_{s=0}^{\infty} A_s Q_n(x) \frac{\mathrm{d}^s y}{\mathrm{d}x^s} + [P_m(x) - aQ_n(x)]y = 0. \tag{7}$$

† Note that when the function $F(z)$ is not a polynomial (in which case we are interested in the present paper), then any equation of the type (1) is considered in literature as a differential equation of infinite order but we show that such an equation does not always have infinitely many linearly independent solutions, i.e. it may be a differential equation of finite order (see also Dimitrov 1981).

On the basis of the equation (7), we determine the linear differential form (see, for instance, Kamke 1959, part 1, ch 5)

$$M_\tau(u)=\sum_{\mu=0}^{m} a_\mu \frac{\mathrm{d}^\mu u(\tau)}{\mathrm{d}\tau^\mu}+[F(\tau)-a]\sum_{\nu=0}^{n} b_\nu \frac{\mathrm{d}^\nu u(\tau)}{\mathrm{d}\tau^\nu}. \tag{8}$$

Then we have

$$M_\tau(\mathrm{e}^{x\tau})=L_x(\mathrm{e}^{x\tau}). \tag{9}$$

Substituting (3) in (7), and taking into account (6), (8) and (9), we obtain

$$\int_C \mathrm{d}\tau\, \mathrm{e}^{x\tau} M_\tau^*(\chi)+\int_C \mathrm{d}\tau \frac{\mathrm{d}V(x,\tau)}{\mathrm{d}\tau}=0 \tag{10}$$

where $M_\tau^*(\chi)$ is the differential form conjugate to $M_\tau(u)$, and

$$V(x,\tau)=\mathrm{e}^{x\tau}\left(\sum_{\mu=0}^{m-1} a_{\mu+1}\sum_{p+q=\mu}(-1)^p x^q \frac{\mathrm{d}^p\chi}{\mathrm{d}\tau^p}+\sum_{\nu=0}^{n-1} b_{\nu+1}\sum_{p+q=\nu}(-1)^p x^q \frac{\mathrm{d}^p}{\mathrm{d}\tau^p}[(F(\tau)-a)\chi]\right). \tag{11}$$

Now it is clear from (10) that the function (3) will be a solution of equation (1) when the amplitude function $\chi(\tau)$ satisfies the following Laplace transform equation in G†

$$M_\tau^*(\chi)\equiv\sum_{\mu=0}^{m}(-1)^\mu a_\mu \frac{\mathrm{d}^\mu\chi}{\mathrm{d}\tau^\mu}+\sum_{\nu=0}^{n}(-1)^\nu b_\nu \frac{\mathrm{d}^\nu}{\mathrm{d}\tau^\nu}\{[F(\tau)-a]\chi\}=0 \tag{12}$$

and the path of integration C is chosen so that the function $V(x,\tau)$ defined by (11) should return to its initial value having passed along it. In this way, the problem of solving differential equation (1) is reduced to the problem of solving differential equation (12) which is of finite order, equal to $\max(m,n)$, and after that to pick out all suitable paths of integration in the formula (3) so that as a result of integration over each of them a linearly independent solution of the equation (1) should be obtained.

In the case (ii) we shall briefly consider only the problem for finding the eigenvalues a and their corresponding eigenfunctions y_a of the operator $F(q_l(x))$ under given boundary conditions. It is clear that when, regardless of the order of the operator L_x, these boundary conditions are related only to values of the function y and its derivatives of order not greater than $(l-1)$, then the eigenfunctions of the differential operator $q_l(x)$ for its respective eigenvalues λ, might be considered as eigenfunctions y_a, i.e.

$$q_l(x)y_a(x)=\lambda y_a(x). \tag{13}$$

Moreover, for the corresponding eigenvalues a, from (1) (when $f(x)=0$), (6) and (13) we obtain

$$a=F(\lambda). \tag{14}$$

† Note that if the path of integration C is a closed curve, then instead of zero in the right-hand side of the equation (12), more generally (in accordance with the Cauchy integral theorem) we may write an arbitrary given function $w(\tau)$ which is holomorphic on C and in the region surrounded by C (on the Riemann surface of the integrand in (3)). By using such a function $w(\tau)$, however, it is possible to avoid the appearance of the Dirac delta function at intermediate stages of the solving of some differential equations.

For example, in the case $l=1$ and when boundary conditions are not imposed, we find from (2) and (13)

$$y_a = \text{constant}\exp\left[\frac{1}{\alpha}\left(\lambda x - \int_0^x \mathrm{d}x\, g(x)\right)\right] \tag{15}$$

for possible values of λ all complex numbers. When the solutions of the problem concern a given finite interval $I=[x_1, x_2]$ under boundary conditions of the form (see, for instance, Kamke 1959, part 2, ch 3)

$$y_a(x_1) = \gamma_0 y_a(x_2)$$

where γ_0 is a constant, then there exists an infinite discrete set of complex eigenvalues λ, determined by the formula

$$\lambda = \lambda_k = \frac{1}{x_2 - x_1}\left[\int_{x_1}^{x_2} \mathrm{d}x\, g(x) - \alpha \ln \gamma_0 + 2\pi i \alpha k\right] \qquad k = 0, \pm 1, \pm 2, \ldots .$$

Therefore, the eigenvalues a and their respective eigenfunctions y_a in this case, in view of (14) and (15), will be

$$a = a_k = F(\lambda_k) \qquad y_a = y_{a_k} = \text{constant}\exp\left[\frac{1}{\alpha}\left(\lambda_k x - \int_0^x \mathrm{d}x\, g(x)\right)\right]$$

$$k = 0, \pm 1, \pm 2, \ldots .$$

Note that in general the exposed Laplace method is applicable also for solving a class of non-homogeneous linear differential equations (including a corresponding class of partial differential equations), and, in particular, the problem for the eigenfunctions in case (ii) is solved in an analogous way when we have a more complicated differential operator $q_l(x)$ of finite order l instead of the type (2). So, the solutions of a non-homogeneous differential equation, which is obtained from (1) by putting a non-homogeneous term $h(x)$ on the right-hand side, in case (i) are also defined by the formula (3), where the amplitude function $\chi(\tau)$ is a solution of the equation (12), but now the path of integration C must be chosen so that for the function (11) the condition

$$\int_C \mathrm{d}\tau \frac{\mathrm{d}V(x, \tau)}{\mathrm{d}\tau} = h(x)$$

would be satisfied.

3. Solutions of some particular differential equations

Now we shall seek the solutions of several operator non-polynomial differential equations of type (1). First we shall consider the equation

$$\left(\alpha\frac{\mathrm{d}}{\mathrm{d}x} + 1\right)^{-1} y + (\beta x - a)y = 0 \tag{16}$$

where α is a real positive constant, while β and a are complex constants in general. In this case the equation (12) assumes the form

$$\beta\frac{\mathrm{d}\chi}{\mathrm{d}\tau} + \left(a - \frac{1}{\alpha\tau + 1}\right)\chi = 0 \tag{17}$$

whence we obtain

$$\chi(\tau) = \text{constant}(\alpha\tau + 1)^{1/\alpha\beta} \exp(-a\tau/\beta). \tag{18}$$

Thus, from (3) and (18), for the solution of the equation (16) in the integral form, when $x \in I = (-\infty, \infty)$, we find

$$y = \text{constant} \int_C \mathrm{d}\tau(\alpha\tau + 1)^{1/\alpha\beta} \exp[(x - a/\beta)\tau] \tag{19}$$

where the path of integration C should be chosen so that the function $V_1(x, \tau) = \chi(\tau)\,\mathrm{e}^{x\tau}$, defined by (11), should return to its initial value having passed along it. For example, if $0 < \operatorname{Re}\beta$, then as a path of integration C, for $x < \operatorname{Re}(a/\beta)$, one can choose the part of the real axis of the complex τ plane defined by $-\alpha^{-1} \leqslant \operatorname{Re}\tau < \infty$, while for $\operatorname{Re}(a/\beta) < x$, the remaining part of the same axis for which $-\infty < \operatorname{Re}\tau \leqslant -\alpha^{-1}$, and in both cases we find from (19)

$$y = \text{constant}\,\mathrm{e}^{-x/\alpha}\{(x - a/\beta)\operatorname{sgn}[x - \operatorname{Re}(a/\beta)]\}^{-1-1/\alpha\beta} \qquad x \neq \operatorname{Re}(a/\beta).$$

Note that in the case $\beta = 0$ we have from (17)

$$\chi(\tau) = \text{constant}\,\delta\left(a - \frac{1}{\alpha\tau + 1}\right) \tag{20}$$

where $\delta(z)$ is the Dirac delta function of a complex variable z. Then, using the representation

$$\delta(z) = \frac{1}{2\pi\mathrm{i}} \lim_{\varepsilon \to 0}\left(\frac{1}{z - \mathrm{i}\varepsilon} - \frac{1}{z + \mathrm{i}\varepsilon}\right) \tag{21}$$

and choosing for C an arbitrary closed curve surrounding only one of the two poles of the function (21) for $z = a - 1/(\alpha\tau + 1)$ (so that the other pole will be outside C), it is obtained right away from (3), (20) and (21)

$$y = \text{constant} \exp\left(\frac{1 - a}{\alpha a} x\right).$$

As a second example we shall consider the equation

$$\left(1 + \alpha\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} y - \left(\frac{\gamma}{|x|} + a\right) y = 0 \qquad x \neq 0 \tag{22}$$

where for simplicity it will be assumed that α, a and γ are real constants, and the sign (i.e. the branch) of the square root denotes its absolute eigenvalues. This equation will be rewritten in the form

$$\pm x\left[\left(1 + \alpha\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^{1/2} - a\right] y - \gamma y = 0 \tag{23}$$

where the upper (lower) sign in front of x is to be used if $0 < x\,(x < 0)$. Moreover, for equation (23), the function (11) and the equation (12) become

$$V_2(x, \tau) = \mathrm{e}^{x\tau}(\sqrt{1 + \alpha\tau^2} - a)\chi(\tau) \tag{24}$$

and

$$\pm(\sqrt{1 + \alpha\tau^2} - a)\frac{\mathrm{d}\chi}{\mathrm{d}\tau} + \left(\gamma \pm \frac{\alpha\tau}{\sqrt{1 + \alpha\tau^2}}\right)\chi = 0. \tag{25}$$

The solution of the equation (25) is as follows:

$$\chi=\frac{\text{constant}}{\sqrt{1+\alpha\tau^2}-a}\left[\left(\frac{1+\mathrm{i}\left(\frac{1+\mathrm{i}\tau\sqrt{\alpha}}{1-\mathrm{i}\tau\sqrt{\alpha}}\right)^{1/2}}{1-\mathrm{i}\left(\frac{1+\mathrm{i}\tau\sqrt{\alpha}}{1-\mathrm{i}\tau\sqrt{\alpha}}\right)^{1/2}}\right)\left(\frac{1-a\left(\frac{1+\mathrm{i}\tau\sqrt{\alpha}}{1-\mathrm{i}\tau\sqrt{\alpha}}\right)^{1/2}+\sqrt{1-a^2}}{1-a\left(\frac{1+\mathrm{i}\tau\sqrt{\alpha}}{1-\mathrm{i}\tau\sqrt{\alpha}}\right)^{1/2}-\sqrt{1-a^2}}\right)^{\mathrm{i}a/\sqrt{1-a^2}}\right]^{\mp\gamma/\sqrt{\alpha}} \tag{26}$$

In this way, the two linearly independent solutions of equation (22) for $x\in I=(-\infty,\infty)$ can be found by substituting the function (26) in the integral formula (3) and by choosing the path of integration C so that, having passed along it, the function (24) for a function χ from (26) should return to its initial value. The solutions, however, will be obtained in a more convenient form by accomplishing first the substitution

$$y=x\eta(x) \tag{27}$$

in the equation (22) and after that, applying the Laplace method, we find the solutions of the obtained equation for the function η, namely

$$\left(x\sqrt{1+\alpha\frac{\mathrm{d}^2}{\mathrm{d}x^2}}+\frac{\alpha}{\sqrt{1+\alpha\,\mathrm{d}^2/\mathrm{d}x^2}}\frac{\mathrm{d}}{\mathrm{d}x}\mp\gamma-ax\right)\eta=0. \tag{28}$$

For this purpose, we shall write

$$\eta=\int_{C'}\mathrm{d}\tau\,\mathrm{e}^{x\tau}\chi_1(\tau) \tag{29}$$

instead of (3). With the help of the Laplace transformation (29) from (28) we obtain the following equation for the function χ_1:

$$(\sqrt{1+\alpha\tau^2}-a)\frac{\mathrm{d}\chi_1}{\mathrm{d}\tau}\pm\gamma\chi_1=0. \tag{30}$$

Besides, the condition for the choice of the path of integration C' in (29) is reduced to the requirement of the function

$$V_2'(x,\tau)=\mathrm{e}^{x\tau}(\sqrt{1+\alpha\tau^2}-a)\chi_1(\tau) \tag{31}$$

to return to its initial value when it passes along it. From (30) we find for the function χ_1,

$$\chi_1=\text{constant}\left(\frac{1-a\left(\frac{1+\mathrm{i}\tau\sqrt{\alpha}}{1-\mathrm{i}\tau\sqrt{\alpha}}\right)^{1/2}+\sqrt{1-a^2}}{1-a\left(\frac{1+\mathrm{i}\tau\sqrt{\alpha}}{1-\mathrm{i}\tau\sqrt{\alpha}}\right)^{1/2}-\sqrt{1-a^2}}\right)^{\mp\mathrm{i}\gamma a/\sqrt{\alpha(1-a^2)}}$$
$$\times\exp\left[\pm\frac{2\mathrm{i}\gamma}{\sqrt{a}}\tan^{-1}\left(\frac{1+\mathrm{i}\tau\sqrt{\alpha}}{1-\mathrm{i}\tau\sqrt{\alpha}}\right)^{1/2}\right]. \tag{32}$$

Thus, for the solutions of the equation (22) for $x \in I = (-\infty, \infty)$, in view of (27), (29) and (32), we get

$$y = \text{constant } x \int_{C'} \mathrm{d}\tau \left(\frac{1 - a\left(\frac{1+\mathrm{i}\tau\sqrt{\alpha}}{1-\mathrm{i}\tau\sqrt{\alpha}}\right)^{1/2} + \sqrt{1-a^2}}{1 - a\left(\frac{1+\mathrm{i}\tau\sqrt{\alpha}}{1-\mathrm{i}\tau\sqrt{\alpha}}\right)^{1/2} - \sqrt{1-a^2}} \right)^{\mp \mathrm{i}\gamma a/\sqrt{\alpha(1-a^2)}}$$
$$\times \exp\left[x\tau \pm \frac{2\mathrm{i}\gamma}{\sqrt{\alpha}} \tan^{-1}\left(\frac{1+\mathrm{i}\tau\sqrt{\alpha}}{1-\mathrm{i}\tau\sqrt{\alpha}}\right)^{1/2} \right]. \tag{33}$$

Here it is to be noted that because of the second order of the differential operator $\sqrt{1+\alpha\,\mathrm{d}^2/\mathrm{d}x^2}$ (Dimitrov 1981), by suitable choices of C' in (33) (or of C in (3) when $\chi(\tau)$ is given by (26)), it is possible to obtain two and not more than two linearly independent solutions of the equation (22).

As an illustration, for $\alpha = -1$ and $0 < \gamma$ we shall obtain the eigenvalues a and their respective (bounded and one-valued continuous together with their first derivatives) eigenfunctions y_a of the operator $\sqrt{1-\mathrm{d}^2/\mathrm{d}x^2} - \gamma/|x|$ under boundary conditions $y_a(\pm\infty) = 0$. In this case, we have from (33)

$$y = \text{constant } x \int_{C'} \mathrm{d}\tau \left(\frac{1 - a[(1-\tau)/(1+\tau)]^{1/2} + \sqrt{1-a^2}}{1 - a[(1-\tau)/(1+\tau)]^{1/2} - \sqrt{1-a^2}} \right)^{\mp\gamma a/\sqrt{1-a^2}}$$
$$\times \exp\{x\tau \pm 2\gamma \tan^{-1}[(1-\tau)/(1+\tau)]^{1/2}\}. \tag{34}$$

From (30), (32) and (34) it is clear, for example, that when $a^2 < 1$ and $0 < ax$, the path of integration C' can be chosen with origin at the point $\tau = \tau_- = \sqrt{1-a^2}$ and after going round the point $\tau = \tau_+ = -\sqrt{1-a^2}$, it would return back to its initial point τ_-, while for $ax < 0$, this can be done in the same way, but with an interchanged role of the points $\tau = \tau_+$ and $\tau = \tau_-$. Note that the integrand expressions in (33) and (34) are, in general, multivalued functions, hence, it is conditionally assumed for them that the powers of the complex magnitudes at each point are chosen to have arguments, smallest in absolute value. In the case we are concerned with, obviously, it is necessary to have

$$\gamma a/\sqrt{1-a^2} = n \qquad n = 1, 2, 3, \ldots \tag{35}$$

in order to satisfy the conditions for the eigenfunctions mentioned above (for $a^2 < 1$). In this way, the eigenvalues a will be

$$a = a_n = n(\gamma^2 + n^2)^{-1/2} \qquad n = 1, 2, 3, \ldots \tag{36}$$

while we find from (34) and (35) for their corresponding eigenfunctions $y_a = y_n$ with the help of the Cauchy integral formula

$$y = y_n = \text{constant } x \lim_{\tau\to\tau_\pm} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\tau^{n-1}} \left[(\tau-\tau_\pm)^n \left(\frac{1 - a_n[(1-\tau)/(1+\tau)]^{1/2} + \sqrt{1-a_n^2}}{1 - a_n[(1-\tau)/(1+\tau)]^{1/2} - \sqrt{1-a_n^2}} \right)^{\mp n} \right.$$
$$\left. \times \exp\{x\tau \pm 2\gamma \tan^{-1}[(1-\tau)/(1+\tau)]^{1/2}\} \right]. \tag{37}$$

In particular, for $n = 1$, from (36) and (37), we have

$$a_1 = \frac{1}{\sqrt{\gamma^2+1}} \qquad y_1 = \text{constant } x \exp\left(-\frac{\gamma|x|}{\sqrt{\gamma^2+1}} \right).$$

As a third example we shall consider the equation

$$F(\mathrm{d}/\mathrm{d}x)y+(\beta x-a)y=0 \tag{38}$$

where a and β are constants. In this case the function (11) and the equation (12) assume the forms, respectively

$$V_3(x,\tau)=\mathrm{e}^{x\tau}\chi(\tau) \tag{39}$$

and

$$\beta\,\mathrm{d}\chi/\mathrm{d}\tau-[F(\tau)-a]\chi=0. \tag{40}$$

Thus, for the solutions of the equation (38) we obtain from (3) and (40)

$$y=\text{constant}\int_C \mathrm{d}\tau\,\exp\left((x-a/\beta)\tau+\frac{1}{\beta}\int_0^\tau \mathrm{d}\tau\,F(\tau)\right) \tag{41}$$

and the choice of the path of integration C must be done so that the function (39), i.e. the integrand in (41) should return to its initial value when it runs along it. Obviously, for fixed values of a and β the solutions (41) are determined from the particular form of the prescribed function $F(\tau)$. Moreover, the already considered equation (16) is a partial case of the equation (38). As another particular example of the equation (38) for $F(\mathrm{d}/\mathrm{d}x)=\mathrm{d}^2/\mathrm{d}x^2$, we find from (41)

$$y=\text{constant}\int_C \mathrm{d}\tau\,\exp[(x-a/\beta)\tau+\tau^3/3\beta] \tag{42}$$

where the path of integration C should start and end in the infinitely far points of these domains of the plane of the complex variable τ, for which $\mathrm{Re}(\tau^3/\beta)<0$. In this way, from (42) only two linearly independent solutions are found, which, with exactness to a constant factor, represent the well known Airy functions $A_i(z)$ and $B_i(z)$ for $z=a/\beta^{2/3}-\beta x$ (see, for instance, Magnus *et al* 1966)

$$A_i(z)=\frac{3^{1/3}}{\pi}\int_0^\infty \mathrm{d}\tau\,\cos(\tfrac{1}{3}\tau^3+z\tau) \qquad B_i(z)=\frac{3^{1/3}}{\pi}\int_0^\infty \mathrm{d}\tau[\sin(\tfrac{1}{3}\tau^3+z\tau)+\exp(z\tau-\tfrac{1}{3}\tau^3)].$$

For $F(\mathrm{d}/\mathrm{d}x)=\exp(\lambda\,\mathrm{d}/\mathrm{d}x)$, where λ is a constant, we obtain from (41) after the substitution of the integration variable τ with $u=\lambda\tau$:

$$y=\text{constant}\int_C \mathrm{d}u\,\exp\left[\left(x-\frac{a}{\beta}\right)\frac{u}{\lambda}+\frac{1}{\lambda\beta}\mathrm{e}^u\right] \tag{43}$$

and, besides, in order that the function (39) should return to its initial value when it runs along the path of integration C, the latter should start and end at points at infinity of the domains of the plane of the complex variable u, for which we have

$$\mathrm{Re}\left(\frac{1}{\lambda\beta}\mathrm{e}^u\right)<0.$$

It is clear that when $\lambda\beta$ is a real negative number, these domains are determined by the inequalities

$$0\leqslant \mathrm{Re}\,u<\infty \qquad (2m-\tfrac{1}{2})\pi<\mathrm{Im}\,u<(2m+\tfrac{1}{2})\pi \qquad m=0,\pm1,\pm2,\ldots \tag{44}$$

while when $\lambda\beta$ is a real positive number, they are determined by the inequalities

$$0\leqslant \mathrm{Re}\,u<\infty \qquad (2m+\tfrac{1}{2})\pi<\mathrm{Im}\,u<(2m+\tfrac{3}{2})\pi \qquad m=0,\pm1,\pm2,\ldots. \tag{45}$$

Hence, in the first case, in view of (44), the path of integration C in (43) can be chosen, for example, so that when it passes along it $\operatorname{Re} u$ should vary from $+\infty$ to 0 for $\operatorname{Im} u = 2\pi m$ $(m \neq 0)$, after that $\operatorname{Im} u$ should vary from $2\pi m$ to 0 for $\operatorname{Re} u = 0$, and finally, $\operatorname{Re} u$ should vary from 0 to $+\infty$ for $\operatorname{Im} u = 0$, while in the second case, in accordance with (45), this can be done in such a way, for instance, that $\operatorname{Re} u$ should vary from $+\infty$ to 0 for $\operatorname{Im} u = (2m+1)\pi$, after that $\operatorname{Im} u$ should vary from $(2m+1)\pi$ to π for $\operatorname{Re} u = 0$, and, finally, $\operatorname{Re} u$ should vary from 0 to $+\infty$ for $\operatorname{Im} u = \pi$. In this way, in both cases, from (43) we obtain a system of infinite number of linearly independent solutions of the equation (38) for an operator $F(\mathrm{d}/\mathrm{d}x) = \exp(\lambda\, \mathrm{d}/\mathrm{d}x)$, which are represented in the following integral forms

$$y = y_m = A_m\left\{\left[1-\exp\left(\frac{2\pi \mathrm{i} m}{\lambda}(x-a/\beta)\right)\right]\int_0^\infty \mathrm{d}u\, \exp\left((x-a/\beta)\frac{u}{\lambda}+\frac{\mathrm{e}^u}{\lambda\beta}\right)\right.$$
$$\left.+\int_0^{2\pi m} \mathrm{d}u\, \exp\left[\frac{\cos u}{\lambda\beta}+\mathrm{i}\left((x-a/\beta)\frac{u}{\lambda}+\frac{\sin u}{\lambda\beta}+\tfrac{1}{2}\pi\right)\right]\right\}$$
$$m = \pm 1, \pm 2, \pm 3, \ldots, \qquad \lambda\beta < 0$$

and

$$y = y_n = B_n\left\{\exp\left[\frac{\pi \mathrm{i}}{\lambda}\left(x-\frac{a}{\beta}\right)\right]\left[1-\exp\left(\frac{2\pi \mathrm{i} n}{\lambda}(x-a/\beta)\right)\right]\int_0^\infty \mathrm{d}u\, \exp\left[\left(x-\frac{a}{\beta}\right)\frac{u}{\lambda}-\frac{\mathrm{e}^u}{\lambda\beta}\right]\right.$$
$$\left.+\int_\pi^{(2n+1)\pi} \mathrm{d}u\, \exp\left[\frac{\cos u}{\lambda\beta}+\mathrm{i}\left((x-a/\beta)\frac{u}{\lambda}+\frac{\sin u}{\lambda\beta}-\tfrac{1}{2}\pi\right)\right]\right\}$$
$$n = \pm 1, \pm 2, \pm 3, \ldots, \qquad 0 < \lambda\beta$$

where A_m and B_n $(m, n = \pm 1, \pm 2, \pm 3, \ldots)$ are arbitrary constants. Therefore, for a transcendental operator $F(\mathrm{d}/\mathrm{d}x) = \exp(\lambda\, \mathrm{d}/\mathrm{d}x)$, in view of the theorem proved in the previous paper of ours (Dimitrov 1981), the differential equation (38) is of infinite order. Naturally, here we could consider the solutions of the equation (38) for many other concrete types of the operator $F(\mathrm{d}/\mathrm{d}x)$, but this will not be done since the method of obtaining these solutions is already entirely clear.

As a fourth example of solving an equation of the type (1), we shall find the solutions of the equation (see the paper of Zhidkov *et al* 1970)

$$[\cosh(\mathrm{i}\, \mathrm{d}/\mathrm{d}x) + U(x) - a]y = 0 \qquad 0 < x \tag{46}$$

where

$$U(x) = \begin{cases} 0 & \text{when } x < \alpha \\ U_0 & \text{when } \alpha < x \end{cases} \tag{47}$$

while a, α and U_0 are real positive constants and $x \neq \alpha$. We find the following equation for the function $\chi(\tau)$ from (12), (46) and (47):

$$[\cos\tau + U_0\theta(x-\alpha) - a]\chi = 0 \tag{48}$$

where

$$\theta(x) = \begin{cases} 0 & \text{when } x < 0 \\ 1 & \text{when } 0 < x. \end{cases}$$

Moreover, for the equation (46) the function (11) is equal to zero. So, we have from (48)

$$\chi = \text{constant}\ \delta(\cos\tau + U_0\theta(x-\alpha) - a) \tag{49}$$

and the path of integration C in (3) might be chosen in consecutive order as an arbitrary closed curve surrounding only one of the two poles of the representation (21) of the Dirac delta function in (49), corresponding to each of the zeros of the argument of that function (so that the other pole will be outside C). Obviously, for the zeros of the argument of the Dirac delta function in (49), we have

$$\tau = \tau_n = \cos^{-1}[a - U_0\theta(x-\alpha)] + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots \qquad x \neq \alpha.$$

In this way, from (3), (21) and (49) it is obtained that the system of fundamental solutions of the equation (46) for $x \in I_1 = (0, \alpha)$ is as follows:

$$y = y_n = \exp[(\cos^{-1} a + 2\pi n)x] \qquad n = 0, \pm 1, \pm 2, \ldots$$

while for $x \in I_2 = (\alpha, \infty)$ it is formed by the functions

$$y = y_n = \exp\{[\cos^{-1}(a - U_0) + 2\pi n]x\} \qquad n = 0, \pm 1, \pm 2, \ldots.$$

Therefore, equation (46) is an ordinary differential equation of infinite order (see also Zhidkov *et al* 1970).

Finally, as a fifth and more special example for solving a differential equation of the type (1) when the operator $q_l(x)$ does not belong to the types (2) we shall consider the equation

$$L_x(y) \equiv \left[1 - \lambda^2\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)\right]^{1/2} y - \frac{\gamma}{x} y = ay \tag{50}$$

where λ, γ and a are constants and $x \in I = (0, \infty)$. The solutions of this equation are also sought to be in the integral form (3), and in a way analogous to that for obtaining equation (12), and taking into account the formula

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^s y = \frac{\mathrm{d}^{2s}y}{\mathrm{d}x^{2s}} + \frac{2s}{x}\frac{\mathrm{d}^{2s-1}y}{\mathrm{d}x^{2s-1}} \qquad s = 1, 2, 3, \ldots,$$

we find that the function $\chi(\tau)$ should satisfy the differential equation

$$\frac{\mathrm{d}\chi}{\mathrm{d}\tau} + \frac{\gamma}{\sqrt{1-\lambda^2\tau^2} - a}\chi = 0 \tag{51}$$

while the choice of the path of integration C should be made so that the function

$$V_5(x, \tau) = \mathrm{e}^{x\tau}(\sqrt{1-\lambda^2\tau^2} - a)\chi(\tau) \tag{52}$$

should return to its initial value having passed along it.

Equation (51) is of the type (30) and hence its solution is found directly from (32) by putting $\alpha = -\lambda^2$ and taking into account only the upper sign in front of γ. However, we shall write the solution of the equation (51) in the form

$$\chi = \text{constant}\left(\frac{\lambda\tau\sqrt{(1-a)/(1+a)} - 1 + \sqrt{1-\lambda^2\tau^2}}{\lambda\tau\sqrt{(1-a)/(1+a)} + 1 - \sqrt{1-\lambda^2\tau^2}}\right)^{\gamma a/\lambda\sqrt{1-a^2}} \times \exp\left[\frac{2\gamma}{\lambda}\tan^{-1}\left(\frac{\sqrt{1-\lambda^2\tau^2} - 1}{\lambda\tau}\right)\right]. \tag{53}$$

In this way, for the solutions of the equation (50), from (3) and (53), after the substitution of the integration variable τ with z, according to the equality

$$\lambda\tau = -2z/(z^2+1) \tag{54}$$

we find

$$y = \text{constant} \int_{C'} dz \frac{z^2-1}{(z^2+1)^2}\left(\frac{z+\sqrt{(1-a)/(1+a)}}{z-\sqrt{(1-a)/(1+a)}}\right)^{\gamma a/\lambda\sqrt{1-a^2}}$$
$$\times \exp\left(-\frac{2xz}{\lambda(z^2+1)}+\frac{2\gamma}{\lambda}\tan^{-1} z\right) \tag{55}$$

where the path of integration C' should be chosen in the plane of the complex variable z so that, in accordance with (52) and (54), the function

$$V'_5(x, z) = \left(\frac{z^2-1}{z^2+1}+a\right)\left(\frac{z+\sqrt{(1-a)/(1+a)}}{z-\sqrt{(1-a)/(1+a)}}\right)^{\gamma a/\lambda\sqrt{1-a^2}}$$
$$\times \exp\left(-\frac{2xz}{\lambda(z^2+1)}+\frac{2\gamma}{\lambda}\tan^{-1} z\right) \tag{56}$$

should return to its initial value having passed along it. When

$$0 < 1 + \mathrm{Re}\left(\frac{\gamma a}{\lambda\sqrt{1-a^2}}\right) \tag{57}$$

for example, we might choose as a path of integration C' in (55) a closed curve which starts from the point $z = -\sqrt{(1-a)(1+a)}$, goes around the point $z = \sqrt{(1-a)/(1+a)}$ and returns to the point $z = -\sqrt{(1-a)/(1+a)}$. After passing along such a curve, the function (56) returns to its initial value of zero. When the condition

$$0 < 1 - \mathrm{Re}\left(\frac{\gamma a}{\lambda\sqrt{1-a^2}}\right) \tag{58}$$

is fulfilled, the path of integration C' in (55) can be chosen in the same way, but with an interchanged role of the points $z = -\sqrt{(1-a)/(1+a)}$ and $z = \sqrt{(1-a)/(1+a)}$. Moreover, remark that the integrand expression in (55) and the function (56) are, in general, multivalued functions of z and therefore it is assumed for them that the powers of the complex magnitudes are always chosen to have arguments smallest in absolute value.

Now we shall briefly consider the problem for finding the eigenvalues a and eigenfunctions y_a of the operator L_x in the equation (50) when λ and γ are real numbers. First, this will be done under the condition that the eigenfunctions and their first derivatives are bounded and single-valued continuous functions and the boundary value condition $y_a(\infty) = 0$ is satisfied. The operator L_x will then be Hermitian and, therefore, its eigenvalues a will be real numbers. From (55) it is seen that in order to satisfy the boundary value condition, we must have

$$\frac{\gamma a}{\lambda\sqrt{1-a^2}} = n \qquad n = 1, 2, 3, \ldots \tag{59}$$

whence we obtain for the eigenvalues a,

$$a = a_n = [1+(\gamma/\lambda n)^2]^{-1/2} \qquad n = 1, 2, 3, \ldots. \tag{60}$$

Equation (59) implies the fulfilment of the condition (57). That is why, when we choose the path of integration C' in the way described above and apply the Cauchy integral formula to the obtained integrals, from (55) for the eigenfunctions $y_a = y_n$, $n = 1, 2, 3, \ldots$, corresponding to the eigenvalues (60), we find

$$y = y_n = C_n \lim_{z \to z_n} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[\frac{z^2 - 1}{(z^2+1)^2} (z + z_n)^n \exp\left(-\frac{2xz}{\lambda(z^2+1)} + \frac{2\gamma}{\lambda} \tan^{-1} z \right) \right] \tag{61}$$

where C_n are arbitrary integration constants, while the magnitudes z_n are determined by the formula

$$z_n = \sqrt{\frac{1-a_n}{1+a_n}} = \left(\frac{\sqrt{1+(\gamma/\lambda n)^2} - 1}{\sqrt{1+(\gamma/\lambda n)^2} + 1} \right)^{1/2} \qquad n = 1, 2, 3, \ldots. \tag{62}$$

In particular, for $n = 1$, we have from (60)–(62)

$$a_1 = \frac{\lambda}{\sqrt{\lambda^2 + \gamma^2}} \qquad y_1 = \text{constant} \exp\left(\frac{-\gamma x}{\lambda \sqrt{\lambda^2 + \gamma^2}} \right).$$

Further, we shall make a remark that the eigenvalues a of the operator L_x in the equation (50) for $1 \leqslant a^2$, form a continuous spectrum. Then the respective eigenfunctions y_a are obtained from (55) for a choice of the path of integration C' according to (57) and (58) so that it should start from the point $z = -\sqrt{(1-a)/(1+a)}$ and after going around the point $z = \sqrt{(1-a)/(1+a)}$, it should return to the point $z = -\sqrt{(1-a)/(1+a)}$, or for a choice of C' in the same way, but with an interchanged role of the points $z = -\sqrt{(1-a)/(1+a)}$ and $z = \sqrt{(1-a)/(1+a)}$. In the general case, the eigenfunctions y_a might be obtained from (55) for a choice of the path of integration C' in the form of a double closed contour so that starting from an arbitrary point between the points $z = -\sqrt{(1-a)/(1+a)}$ and $z = \sqrt{(1-a)/(1+a)}$ in the plane of the complex variable z and in the beginning surrounding these two points in positive direction and after that surrounding both points in negative direction we should return again to the initial point.

In conclusion, we shall note that the integral representation method and, in particular, the Laplace method, as is known, can be used for finding numerical and approximate solutions, which, in view of what was said in the present paper, is also related to the solutions of a relatively wide class of the linear operator non-polynomial differential equations.

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